

# Note on lacunary Fourier series on nonabelian groups

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## Abstract

We show that the classical equivalence between the BMO norm and the  $L^2$  norm of a lacunary Fourier series has an analogue on any discrete group  $G$  equipped with a conditionally negative function.

## Introduction

Throughout this article, we consider a discrete group  $G$  and a conditionally negative length  $\psi$  on  $G$ . That is to say  $\psi$  is a  $\mathbb{R}_+$ -valued function on  $G$  satisfying  $\psi(e) = 0$ ,  $\psi(g) = \psi(g^{-1})$ , and

$$\sum_{g,h} \overline{a_g} a_h \psi(g^{-1}h) \leq 0 \quad (1)$$

for any finite collection of coefficients  $a_g \in \mathbb{C}$  with  $\sum_g a_g = 0$ . We say a sequence  $h_k \in G$  is  $\psi$ -lacunary if there exists a constant  $\delta > 0$  such that

$$\begin{aligned} \psi(h_{k+1}) &\geq (1 + \delta)\psi(h_k) \\ \psi(h_k^{-1}h_{k'}) &\geq \delta \max\{\psi(h_k), \psi(h_{k'})\}. \end{aligned}$$

for any  $k, k'$ . Let  $\lambda$  be the regular left representation of  $G$ . We say

$$x = \sum_k c_k \lambda_{h_k}$$

is a  $\psi$ -lacunary Fourier series if the sequence  $h_k$  is  $\psi$ -lacunary.

When  $G = \mathbb{Z}$ , and  $\psi(k) = |k|$ ,  $k \in \mathbb{Z}$ . Kochneff/Sagher/Zhou ([3]) prove that, for any  $\psi$ -lacunary Fourier series  $x = \sum_k c_k \lambda_{h_k} \in L^2(\mathbb{T})$ , we have

$$\|x\|_{BMO}^2 \simeq \sum_k |c_k|^2. \quad (2)$$

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By interpolation, this implies that every lacunary Fourier series has an equivalent  $L^p$  and  $L^2$  norm, which is a fundamental theory in the classical Fourier analysis. We will show that Kochneff/Sagher/Zhou' result extends to non-abelian discrete groups by considering semigroup BMO associated with  $\psi$ , while an analogue of Rudin's theorem on the size of Sidon sets (thus Lelièvre's theorem on BMO) fails for  $\mathbb{F}_2$ .

## 1 BMO estimate.

Given a discrete group  $G$ , we denote by  $(\mathcal{L}(G), \tau)$  the group von Neumann algebra with its canonical trace  $\tau$ . Denote by  $L^p(\hat{G})$  the associated noncommutative  $L^p$  spaces, that is the closure of  $\mathcal{L}(G)$  w.r.t. the norm  $\|x\|_p = (\tau|x|^p)^{\frac{1}{p}}$ . If  $G$  is abelian, then  $L^p(\hat{G})$  is the canonical  $L^p$  space of functions on the dual group  $\hat{G}$ . In particular, if  $G = \mathbb{Z}$ , then  $\lambda_k = e^{ikt}$ ,  $k \in \mathbb{Z}$  and  $L^p(\hat{\mathbb{Z}}) = L^p(\mathbb{T})$ , the space of all  $p$ -integrable functions on the unit circle. Please refer to [11] for details on noncommutative  $L^p$  spaces.

Given a conditionally negative length  $\psi$  on  $G$ , Schöenberg's theorem says that

$$T_t : \lambda_g = e^{-t\psi(g)} \lambda_g$$

extends to a symmetric Markov semigroups of operators on the group von Neumann algebra  $\mathcal{L}(G)$ . Following [5] and [8], let us set

$$\|x\|_{\text{BMO}_c(\psi)} = \sup_{0 < t < \infty} \|T_t|x - T_t x|^2\|^{\frac{1}{2}}, \quad (3)$$

for  $x \in L^2(\hat{G})$ . Let  $\text{BMO}(\psi)$  be the space of all  $x \in L^2(\hat{G})$  such that

$$\|x\|_{\text{BMO}(\psi)} = \max\{\|x\|_{\text{BMO}_c(\psi)}, \|x^*\|_{\text{BMO}_c(\psi)}\} < \infty. \quad (4)$$

**Lemma 1.** ([JM12]) *We have the following interpolation result*

$$[\text{BMO}(\psi), L_0^1(\hat{G})]_{\frac{1}{p}} = L_0^p(\hat{G})$$

for  $1 < p < \infty$ . Here  $L_0^p(\hat{G}) = L^p(\hat{G})/\ker\psi$ .

**Lemma 2.** *For  $a_k \in \mathbb{R}_+$ ,  $c_k, b_k \in B(H)$ , we have*

$$\left\| \sum_k a_k c_k^* b_k \right\| \leq \left\| \sum_k |c_k|^2 a_k \right\|^{\frac{1}{2}} \left\| \sum_k |b_k|^2 a_k \right\|^{\frac{1}{2}}. \quad (5)$$

*Proof.* This is simply the Cauchy-Schwartz inequality.  $\square$

**Theorem 1.** *Assume  $(h_k)$  is a  $\psi$ -lacunary sequence. Then, for any  $x = \sum_k c_k \lambda_{h_k}$ , we have*

$$\|x\|_{\text{BMO}(\psi)}^2 \simeq^{c_\delta} \max\left\{ \left\| \sum_k |c_k|^2 \right\|, \left\| \sum_k |c_k^*|^2 \right\| \right\}. \quad (6)$$

*Proof.* An easy calculation shows that

$$T_t|x - T_tx|^2 = \sum_{k,j} a_{k,j} (c_k \lambda_{h_k})^* c_j \lambda_{h_j},$$

with

$$a_{k,j} = e^{-t\psi(h_k^{-1}h_j)}(1 - e^{-t\psi(h_k^{-1})})(1 - e^{-t\psi(h_j)}) \geq 0.$$

By the subadditivity of  $\psi$  we have that  $\psi(h_k^{-1}h_j) \geq |\psi(h_k) - \psi(h_j)|$ . So

$$\begin{aligned} \sum_k a_{k,j} &\leq \sum_{t\psi(h_k) \leq 1} (1 - e^{-t\psi(h_k^{-1})}) + \sum_{t\psi(h_k) > 1} e^{-t\psi(h_k^{-1}h_j)} \\ &\leq \sum_{t\psi(h_k) \leq 1} t\psi(h_k) + \sum_{t\psi(h_k) > 1} e^{-t\delta\psi(h_k)} \\ &\leq 1 + \delta^{-1} + \frac{1}{1 - e^{-\delta^2}} \leq c_\delta. \end{aligned}$$

We then get  $\sup_j \sum_k a_{k,j} \leq c_\delta$ . Similarly,  $\sup_k \sum_j a_{k,j} \leq c_\delta$ . By Lemma, we have

$$\begin{aligned} \|T_t|x - T_tx|^2\| &\leq \left\| \sum_{k,j} |c_k|^2 a_{k,j} \right\|^{\frac{1}{2}} \left\| \sum_{k,j} |c_j|^2 a_{k,j} \right\|^{\frac{1}{2}} \\ &\leq c_\delta \left\| \sum_k |c_k|^2 \right\|. \end{aligned}$$

Taking supremum on  $t$ , we get  $\|x\|_{BMO_c}^2 \leq c_\delta \left\| \sum_k |c_k|^2 \right\|$ . Taking the adjoint, we prove the upper estimate. The lower estimate is obvious.  $\square$

Given a length-lacunary sequence  $h_k \in G$ , define the linear map  $T$  from  $L^\infty(\ell_2)$  to  $BMO$  by

$$T((c_k)) = \sum_k c_k \lambda_{h_k}.$$

Then  $T$  has a norm  $c_\delta$  from  $L^\infty(\ell_2)$  to  $BMO$  and norm 1 from  $L^2(\ell_2)$  to  $L^2(\hat{G})$ . By the interpolation result Lemma 1, we get

**Corollary 1.** *Assume  $(h_k)$  is a  $\psi$ -lacunary sequence. We have that, for any  $x = \sum_k c_k \lambda_{h_k}$ ,*

$$\|x\|_p^2 \leq c_\delta^{\frac{p-2}{p}} p^2 \max\left\{ \left\| \sum_k |c_k|^2 \right\|_{\frac{p}{2}}, \left\| \sum_k |c_k^*|^2 \right\|_{\frac{p}{2}} \right\}. \quad (7)$$

**Remark 1.** Corollary 1 is independently proved in [6] by using noncommutative Riesz transforms associated with semigroups. If,  $G = \mathbb{F}_n$ ,  $\psi$  is the reduced word length, it is also easy to verify that  $\psi$ -lacunary set is  $B(2)$  in the sense of W. Rudin, so it is a  $\Lambda_4$  set by Harcharras's work[2]. This does not seems clear for  $B(p)$  with  $p > 2$ .

## 2 Large $\Lambda_\infty$ sets on $\mathbb{F}_2$

We call a subset  $A \in G$  is completely Sidon, if  $\{\lambda_h, h \in A\}$  is completely unconditional in  $\mathcal{L}(\hat{G})$ , i.e. there exists a constant  $C_A$  such that

$$\left\| \sum_{h_k \in A} \varepsilon_k c_k \lambda_{h_k} \right\| \leq C_A \left\| \sum_{h_k \in A} c_k \lambda_{h_k} \right\|,$$

for any choice  $\varepsilon_k = \pm 1$ ,  $c_k \in B(H)$ . We call a subset  $A \in G$  is completely  $\Lambda_\infty$ , if there exists a constant  $C_A$  such that

$$\left\| \sum_{h_k \in A} c_k \lambda_{h_k} \right\| \leq C_A \max \left\{ \left\| \sum_{h_k \in A} |c_k|^2 \right\|^{\frac{1}{2}}, \left\| \sum_{h_k \in A} |c_k^*|^2 \right\|^{\frac{1}{2}} \right\}, \quad (8)$$

for any choice of finite many  $c_k \in B(H)$ . We say  $A$  is completely  $\Lambda_{bmo, \psi}$  if we take the  $BMO(\psi)$ -norm on the left hand side of (8). Obviously, a completely  $\Lambda_\infty$  set is completely  $\Lambda_{bmo, \psi}$  for any  $\psi$ , and is completely Sidon.

Let  $\mathcal{P}_d$  ( $\mathcal{P}_{\leq d}$ ) be the collection of all reduced words of  $\mathbb{F}_n$  with length  $= d$  ( $\leq d$ ). When  $G = \mathbb{Z}$ , a classical theory of Rudin says that, for any Sidon set  $A$  of  $\mathbb{F}_1$ , we have  $\#(A \cap \mathcal{P}_{\leq d}) \lesssim \log \# \mathcal{P}_{\leq d}$ , and Lelièvre ([4]) prove that every  $\Lambda_{bmo, |\cdot|}$  is a finite combination of Hadamard lacunary sets, thus a Sidon set.

Fix a generating set  $S = \{g_k, k \in \mathbb{Z}_*\}$  of  $\mathbb{F}_\infty$ , with the convention that  $g_k^{-1} = g_{-k}$ . Let  $\mathcal{Q}_n \subset \mathbb{F}_\infty$  be the collection of symmetric words of length  $2n$ ,

$$\mathcal{Q}_n = \{g_{k_1} g_{k_2} \cdots g_{k_n} g_{k_n} \cdots g_{k_2} g_{k_1}; |g_{k_j}| = 1, g_{k_j} \neq g_{k_{j+1}}^{-1}, k_j \in \mathbb{Z}_*\}.$$

The following Proposition is the key observation for our example. We include a proof although this maybe obvious for experts.

**Proposition 1.**  $\mathcal{Q}_n$  is a free subset of  $\mathbb{F}_\infty$ .

*Proof.* Let us first introduce a few notations. Given a reduced word  $h \in \mathbb{F}_\infty$ , we denote by  $L_h$  the subset of all reduced words  $g$  that start with  $h$ , that is  $L_h = \{g \in \mathbb{F}_\infty; |g| \geq |h|, |h^{-1}g| = |g| - |h|\}$ . Suppose  $h \in \mathcal{P}_{2n}$ , denote by  $h^l, h^r$  the left half and the right half of  $h$ , i.e. the reduced words in  $\mathcal{P}_n$  such that  $h = h^l h^r$ . We will use the fact, that the condition  $|hg| > |g|$  holds iff  $g \notin L_{(h^r)^{-1}}$  and implies that  $hg \in L_{h^l}$ .

Given any  $m$  elements  $x_j \in \mathcal{Q}_n, 1 \leq j \leq m$  such that  $x_k^{-1} \neq x_{k+1}$  for any  $1 \leq k < m$ , it is obvious that  $|x_2 x_1| > |x_1|$ . Assume  $|x_j \cdots x_2 x_1| > |x_{j-1} \cdots x_2 x_1|$ . That is  $|x_j g| > |g|$  for  $g = x_{j-1} \cdots x_2 x_1$ . Then  $x_j g \in L_{x_j^l}$ . So  $g' = x_j g \notin L_{(x_{j+1}^r)^{-1}}$  since  $x_j \neq x_{j+1}^{-1}$ . Then  $|x_{j+1} g'| > |g'|$ . We then get  $|x_j \cdots x_2 x_1| > |x_{j-1} \cdots x_2 x_1|$  for all  $1 < j \leq m$  by induction. Therefore,  $x_m \cdots x_2 x_1 \neq e$ . We then conclude that  $\mathcal{Q}_n$  is a free set.  $\square$

Now, for  $x = \sum_{h \in \mathcal{Q}_n} c_h \lambda_h$  with  $h \in \mathcal{Q}_n$  and  $c_h \in B(H)$ , we have by Haagerup and Pisier's inequality ([1]) that

$$\|x\| \leq 2 \max \left\{ \|\tau |x|^2\|^{\frac{1}{2}}, \|\tau |x^*|^2\|^{\frac{1}{2}} \right\}. \quad (9)$$

**Example.** Let  $\pi$  be the group homomorphism from  $\mathbb{F}_\infty$  into  $\mathbb{F}_2$  with free generators  $a, b$ , such that

$$\pi(g_k) = a^k b a^{-k}, k \in \mathbb{N}.$$

By (9),  $\pi(\mathcal{Q}_n)$  is a complete  $\Lambda_\infty$  set of  $\mathbb{F}_2$  for each  $n \in \mathbb{N}$ . Therefore, it is completely Sidon, and is completely  $\Lambda_{bmo(|\cdot|)}$  with  $|\cdot|$  the word length on  $\mathbb{F}_2$ . However,  $\pi(\mathcal{Q}_n)$  is not a finite union of  $|\cdot|$ -lacunary set, contrary to Lelièvre's theorem for  $\mathbb{F}_1$ . In fact, it is easy to see that  $\#(\pi(\mathcal{Q}_n) \cap \mathcal{P}_{\leq 2nm}) \simeq m^n$  while  $\log \# \mathcal{P}_{\leq 2nm} \simeq nm$  as  $m \rightarrow \infty$ .

**Remark 2.** Let  $\phi$  be an injection from  $\mathbb{N}$  to  $\mathbb{N}$ . Let  $\phi(k) = -\phi(-k)$  for  $k < 0$ . (9) holds for

$$\mathcal{Q}_n = \{g_{k_1} g_{k_2} \cdots g_{k_n} g_{\phi(k_n)} \cdots g_{\phi(k_2)} g_{\phi(k_1)}; |g_{k_j}| = 1, g_{k_j} \neq g_{k_{j+1}}^{-1}\}$$

as well.

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